BIALGEBROIDS, $\times_A$-BIALGEBRAS AND DUALITY

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Abstract. An equivalence between Lu's bialgebroids, Xu's bialgebroids with an anchor and Takeuchi's $\times_A$-bialgebras is explicitly proven. A new class of examples of bialgebroids is constructed. A (formal) dual of a bialgebroid, termed bicoalgebroid, is defined. A weak Hopf algebra is shown to be an example of such a bicoalgebroid.

1. Introduction

For some time various generalisations of the notion of a bialgebra, in which a bialgebra is required to be a bimodule but not necessarily an algebra over a (noncommutative) ring have been considered. Motivated by the problem of classification of algebras, a definition of a generalised Hopf algebra was first proposed by Sweedler [17] and later generalised by Takeuchi [19]. This was based on a new definition of a tensor product over noncommutative rings, termed the $\times_A$-product. Several years later, motivated by some problems in algebraic topology Ravenel introduced the notion of a commutative Hopf algebroid [13], which is a special case of the Takeuchi construction. With the growing interest in quantum groups, bialgebroids were discussed in the context of noncommutative [11], and Poisson geometry. In the latter case, the most general definitions were given by Lu [8] and Xu [20]. Another generalisations of finite Hopf algebras, termed weak Hopf algebras, appeared in relation to integrable spin chains and classification of subfactors of von Neumann algebras [2] [12]. In [6] weak Hopf algebras have been shown to be examples of Lu's bialgebroids.

The aim of the present paper is threefold. Since there is a number of different definitions of generalised bialgebras, it is important to study what are the relations between these definitions. Thus our first aim (Section 2) is to collect these different definitions and make it clear that the notions of a Takeuchi's $\times_A$-bialgebra, Lu's bialgebroid, and Xu's bialgebroid with an anchor are equivalent to each other (Section 3). Although this fact in itself seems to be not new (cf. [15, p. 273], where the equivalence of the first and second notions is attributed to P. Xu [20]), to the best of our knowledge,

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there is no explicit and complete proof of this equivalence in the literature. Hereby we provide such a proof, and hope that this clarifies some minor misunderstandings in the field (e.g. it seems to be claimed in [20, p. 546] that Lu's bialgebroid is equivalent to Xu's bialgebroid *without an anchor*). Our second aim (Section 4) is to construct new examples of bialgebroids. We show how to associate a bialgebroid to a braided commutative algebra in the category of Yetter-Drinfeld modules. This result generalises an example considered by Lu. In fact we show that the smash product of a Hopf algebra with an algebra in the Yetter-Drinfeld category is a bialgebroid if and only if the algebra is braided commutative. In particular, a bialgebroid over braided symmetric algebra $S_R(n)$ is associated to any solution of the quantum Yang-Baxter equation $R \in M_n(k) \otimes M_n(k)$. Our third aim (Section 5) is to propose a notion that is dual to a bialgebroid. We term such an object a *bicoalgebroid*. It is well-known that a bialgebra is a self-dual notion in the following sense. The axioms of a bialgebra are invariant under formal reversing of the arrows in the commutative diagrams that constitute the definition of a bialgebra. In the case of a bialgebroid such a formal operation on commutative diagrams produces a new object. We belive that this object will play an important role in constructing a self-dual generalisation of a bialgebra which should involve both a bialgebroid and a bicoalgebroid.

2. Preliminaries

2.1. **Notations.** All rings in this paper have 1, a ring map is assumed to respect 1, and all modules over a ring are assumed to be unital. For a ring A, $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}$) denotes the category of right (resp. left) A-modules, and ${}_A\mathcal{M}_A$ denotes the category of (A, A)-bimodules. The action of A is denoted by a dot between elements.

Throughout the paper k denotes a commutative ring. Unadorned tensor product is over k. For a k-algebra A we use m_A to denote the product as a map and 1_A to denote unit both as an element of A and as a map $k \to A$, $\alpha \to \alpha 1$. $\mathrm{End}(A)$ denotes the algebra of k-linear endomorphisms of A. For a k-coalgebra C we use Δ to denote the coproduct, ϵ to denote the counit; $\mathcal{M}^C$ will be the category of right C-comodules. We use the Sweedler notation, i.e. $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for coproducts, and $\rho^M(m) = m_{<0>} \otimes m_{<1>}$ for coactions (summation understood).

Let A be a k-algebra. Recall from [16] that an *A-coring* is an (A,A)-bimodule $\mathcal{C}$ together with (A,A)-bimodule maps $\Delta_{\mathcal{C}}:\mathcal{C}\to\mathcal{C}\otimes_A\mathcal{C}$ called a coproduct and $\epsilon_{\mathcal{C}}:\mathcal{C}\to A$ called a counit, such that

$$(\Delta_{\mathcal{C}}\otimes_A\mathcal{C})\circ\Delta_{\mathcal{C}}=(\mathcal{C}\otimes_A\Delta_{\mathcal{C}})\circ\Delta_{\mathcal{C}},\quad (\epsilon_{\mathcal{C}}\otimes_A\mathcal{C})\circ\Delta_{\mathcal{C}}=(\mathcal{C}\otimes_A\epsilon_{\mathcal{C}})\circ\Delta_{\mathcal{C}}=\mathcal{C}.$$

Let R be a k-algebra. Recall from [17], [19] that an *R-ring* or an *algebra over R* is a pair (U,i), where U is a k-algebra and $i:R\to U$ is an algebra map. If (U,i) is an R-ring then U is an (R,R)-bimodule with the structure provided by the map i, $r\cdot u\cdot r':=i(r)ui(r')$. A map of R-rings $f:(U,i)\to(V,j)$ is a k-algebra map $f:U\to V$ such that $f\circ i=j$. Equivalently, a map of R-rings is a k-algebra map that is a left or right R-module map. Indeed, clearly if $f:(U,i)\to(V,j)$ is a map of R-rings it is an algebra and R-bimodule map. Conversely, if f is a left R-linear algebra map then for all $r\in R$, $f(i(r))=f(r\cdot 1_U)=r\cdot f(1_U)=j(r)$, and similarly in the right R-linear case.

2.2. **Algebras over enveloping algebras: A^e-rings.** Let A be an algebra and $\bar{A}=A^{op}$ the opposite algebra. For $a\in A$, $\bar{a}\in\bar{A}$ is the same a but now viewed as an element in $\bar{A}$, i.e. $a\to\bar{a}$ is an antiisomorphism of algebras. Let $A^e=A\otimes\bar{A}$ be the enveloping algebra of A. Note that a pair (H,i) is an A^e-ring if and only if there exist an algebra map $s:A\to H$ and an anti-algebra map $t:A\to H$, such that $s(a)t(b)=t(b)s(a)$, for all $a,b\in A$. Explicitly, $s(a)=i(a\otimes 1)$ and $t(a)=i(1\otimes\bar{a})$, and, conversely, $i(a\otimes\bar{b})=s(a)t(b)$. This simple observation shows that the bialgebroids of Lu [8] (cf. Section 2.4 below) and $\times_A$-bialgebras of Takeuchi [19] (cf. Section 2.5 below) have the same input data.

In the sequel, the expression "let (H,s,t) be an A^e-ring" will be understood to mean an algebra H with algebra maps $s,t:A\to H$ as described above. A is called a *base algebra*, H a *total algebra*, s the *source* map and t the *target* map.

A standard example of an A^e-ring is provided by $\mathrm{End}(A)$. In this case $i:A\otimes\bar{A}\to\mathrm{End}(A)$, $i(a\otimes\bar{b})(x)=axb$. The source and the target come out as, $s(a)(x)=ax$ and $t(b)(x)=xb$. It follows that $\mathrm{End}(A)$ is an A^e-bimodule via i. In particular, $\mathrm{End}(A)$ is a left A^e-module via

$$(a\cdot f)(b)=af(b),\qquad (\bar{a}\cdot f)(b)=f(b)\bar{a},\tag{1}$$

for all $a, b \in A$ and $f \in \mathrm{End}(A)$.

As explained at the end of Section 2.1, an A^e-ring (H, s, t) is an A^e-bimodule. This means that H is an A-bimodule with explicit actions, $a \cdot h \cdot b = s(a)hs(b)$, and also that it is an $\bar{A}$-bimodule with actions $\bar{a} \cdot h \cdot \bar{b} = t(a)ht(b)$, for all $a, b \in A$ and $h \in H$.

2.3. **The key A^e-ring associated to an A^e-ring.** Let (H, s, t) be an A^e-ring and view H as an A-bimodule, with the left A-action given by the source map s, and the right A-action which descends from the left $\bar{A}$-action given by the target map t, i.e.,

$$a \cdot h = s(a)h, \qquad h \cdot a = t(a)h, \qquad \forall a \in A, h \in H. \tag{2}$$

Consider an Abelian group $H \otimes_A H$ which is an A-bimodule via the following actions $a \cdot' (g \otimes_A h) = gt(a) \otimes_A h$, and $(g \otimes_A h) \cdot' a = g \otimes_A hs(a)$, for all $a \in A$ and $g \otimes_A h \in H \otimes_A H$. Define $\Gamma = \Gamma(H, s, t) := (H \otimes_A H)^A$, i.e.,

$$\Gamma = \{g = \sum g^1 \otimes_A g^2 \in H \otimes_A H \mid \forall a \in A,\ \sum g^1 t(a) \otimes_A g^2 = \sum g^1 \otimes_A g^2 s(a)\}.$$

The next proposition can be proved directly (cf. [19, Proposition 3.1]).

Proposition 2.1. *Let (H, s, t) be an A^e-ring. Then $\Gamma = \Gamma(H, s, t)$ is an A^e-ring with the algebra structure $(\sum g^1 \otimes_A g^2)(\sum h^1 \otimes_A h^2) = \sum g^1 h^1 \otimes_A g^2 h^2$, the unit $1_H \otimes_A 1_H$ and the algebra map $i : A \otimes \bar{A} \to \Gamma$, $a \otimes_A \bar{b} \mapsto s(a) \otimes_A t(b)$.*

2.4. **Bialgebroids.** Let (H, s, t) be an A^e-ring and view $H \in {}_A\mathcal{M}_A$ using the actions in equations (2). Also, view $H \otimes_A H \in {}_A\mathcal{M}_A$ with the natural actions $a \cdot (g \otimes_A h) \cdot b = s(a)g \otimes_A t(b)h$.

Definition 2.2 ([8])**.** Let (H, s, t) be an A^e-ring. We say that $(H, s, t, \Delta, \epsilon)$ is an *A-bialgebroid* iff

(B1) (H, Δ, ϵ) is an A-coring;

(B2) $\mathrm{Im}(\Delta) \subseteq \Gamma(H, s, t)$ and the corestriction of the coproduct $\Delta : H \to \Gamma(H, s, t)$ is an algebra map;

(B3) $\epsilon(1_H) = 1_A$ and for all $g,\ h \in H$

$$\epsilon(gh) = \epsilon\Big(gs(\epsilon(h))\Big) = \epsilon\Big(gt(\epsilon(h))\Big). \tag{3}$$

An *antipode* for an A-bialgebroid H is an anti-algebra map $\tau : H \to H$ such that

(ANT1) $\tau \circ t = s$;

(ANT2) $m_H \circ (\tau \otimes H) \circ \Delta = t \circ \epsilon \circ \tau$;

(ANT3) There exists a section $\gamma : H \otimes_A H \to H \otimes H$ of the natural projection $H \otimes H \to H \otimes_A H$ such that $m_H \circ (H \otimes \tau) \circ \gamma \circ \Delta = s \circ \epsilon$. An A-bialgebroid with an antipode is called a *Hopf algebroid*.

The notion of a bialgebroid was introduced by J.-H. Lu in [8]. Condition (B2), in its present form, was first stated by P. Xu [20], while (B3) in this form appears in [18] (in a slightly different convention though).

Remark 2.3. 1) As explained in Section 2.1 the axiom (B1) requires that $\Delta : H \to H \otimes_A H$ and $\epsilon : H \to A$ are maps in ${}_A\mathcal{M}_A$, Δ is coassociative and ϵ is a counit for Δ. The counit property explicitly means that for all $h \in H$,

$$s\Big(\epsilon(h_{(1)})\Big)h_{(2)} = t\Big(\epsilon(h_{(2)})\Big)h_{(1)} = h. \tag{4}$$

The first condition of (B2) explicitly means that $h_{(1)}t(a) \otimes_A h_{(2)} = h_{(1)} \otimes_A h_{(2)}s(a)$, for all $a \in A$ and $h \in H$.

2) Definition 2.2 is equivalent to [8, Definition 2.1]. Indeed, in [20, Proposition 3.2] it is shown that (B2) in Definition 2.2 is equivalent to condition 4 in [8, Definition 2.1], which states that the kernel of the map $\Phi : H \otimes H \otimes H \to H \otimes_A H$, $\Phi(g \otimes h \otimes l) = \Delta(g)(h \otimes l)$ is a left ideal of $H \otimes \bar{H} \otimes \bar{H}$.

On the other hand condition (3) of (B3) is equivalent to the condition in [8] that $\ker(\epsilon)$ is a left ideal in H. Indeed, suppose that $\ker(\epsilon)$ is a left ideal in H. Then using that ϵ is an A-bimodule map and that $\epsilon(1) = 1$ one obtains that $h - s(\epsilon(h)), h - t(\epsilon(h)) \in \ker(\epsilon)$, so (3) holds. The converse is obvious.

3) The facts that ϵ preserves unit and is an A-bimodule map imply that s and t are sections of ϵ, i.e., $\epsilon(s(a)) = \epsilon(t(a)) = a$ for all $a \in A$. Using this fact and (3) one easily finds that $\epsilon(gs(a)) = \epsilon(gt(a))$, for all $a \in A$ and $g \in H$ Similarly, the facts that Δ is a unital and A-bimodule map imply that

$$\Delta(s(a)) = s(a) \otimes_A 1_H, \qquad \Delta(t(a)) = 1_H \otimes_A t(a). \tag{5}$$

4) For an A^e-ring (H, s, t), let $F : {}_H\mathcal{M} \to {}_A\mathcal{M}_A$, be the restriction of scalars functor. The actions of A on H are given by equations (2). If $(H, s, t, \Delta, \epsilon)$ is an A-bialgebroid, then ${}_H\mathcal{M}$ has a monoidal structure such that F is a strict monoidal functor. For all $M, N \in {}_H\mathcal{M}$, the tensor product $M \otimes_A N$ is in ${}_H\mathcal{M}$ via $h \cdot (m \otimes_A n) = h_{(1)} \cdot m \otimes_A h_{(2)} \cdot n$.

The right hand side is well defined because $\mathrm{Im}(\Delta) \subseteq \Gamma$. A is the unit object, when viewed in ${}_H\mathcal{M}$ via the action $h \triangleright a = \epsilon(hs(a)) = \epsilon(ht(a))$, for all $h \in H$, $a \in A$. The fact that this is an action follows from (3) (cf. [8, Eq. (8)]).

Next we recall the notion of a bialgebroid with an anchor from [20]. For a k-algebra A, $\mathrm{End}(A)$ is an A^e-ring as described in Section 2.2. In fact, we are interested only in the structure of $\mathrm{End}(A)$ as a left $A \otimes \bar{A}$-module given by equations (1). When, following [20], $\mathrm{End}(A)$ is viewed as an A-bimodule, the structure maps are

$$(a \cdot f)(b) = af(b), \quad (f \cdot a)(b) = f(b)a, \tag{6}$$

for all a, $b \in A$ and $f \in \mathrm{End}(A)$. As before the total algebra H of an A^e-ring (H, s, t), is viewed as an A-bimodule by equations (2).

Leaving aside the redundant part of [20, Definition 3.4] that arises from condition ii) there, [20, Proposition 3.3], and from the fact that a counit of a coring is unique the following definition is the same as [20, Definition 3.4]

Definition 2.4. Let (H, s, t) be an A^e-ring. (H, s, t, Δ, μ) is called an *A-bialgebroid with an anchor* μ if

(BA1) $\Delta : H \to H \otimes_A H$ is a coassociative A-bimodule map;

(BA2) $\mathrm{Im}(\Delta) \subseteq \Gamma(H, s, t)$ and its corestriction $\Delta : H \to \Gamma(H, s, t)$ is an algebra map;

(BA3) $\mu : H \to \mathrm{End}(A)$ is an algebra and an A-bimodule map such that

(A1) $s(h_{(1)} \triangleright a)h_{(2)} = hs(a)$,

(A2) $t(h_{(2)} \triangleright a)h_{(1)} = ht(a)$,

where $\mu(h)(a) = h \triangleright a$, for all $a \in A$ and $h \in H$.

Note that the left hand sides of (A1) and (A2) are well defined since μ is an A-bimodule map, i.e., for all a, $b \in A$ and $h \in H$

$$\mu(s(a)h)(b) = a\mu(h)(b), \qquad \mu(t(a)h)(b) = \mu(h)(b)a. \tag{7}$$

2.5. **$\times_A$-bialgebras.** The notion of a $\times_A$-bialgebra was first introduced by M.E. Sweedler [17] for a commutative A and then generalised by M. Takeuchi [19] to an arbitrary A. In this section we briefly recall Takeuchi's definition (see [19] for details).

Let M and N be A^e-bimodules. Following MacLane, let

$$\int_a {}_{\bar{a}}M \otimes {}_aN := M \otimes N / < \{\bar{a}m \otimes n - m \otimes an | \forall a \in A\} >,$$

$$\int^b M_{\bar{b}} \otimes N_b := \{\sum_i m_i \otimes n_i \in M \otimes N | \forall b \in A,\ \sum_i m_i \bar{b} \otimes n_i = \sum m_i \otimes n_i b\}.$$

Define,

$$M \times_A N := \int^b \int_a {}_{\bar{a}}M_{\bar{b}} \otimes {}_aN_b.$$

The operation $- \times_A - : {}_{A^e}\mathcal{M}_{A^e} \times {}_{A^e}\mathcal{M}_{A^e} \to {}_{A^e}\mathcal{M}_{A^e}$ is a bifunctor. Here, for $M, N \in {}_{A^e}\mathcal{M}_{A^e}$, the product $M \times_A N$ is in ${}_{A^e}\mathcal{M}_{A^e}$ with the actions given by

$$(a \otimes \bar{a}) \cdot (\sum_i m_i \otimes_A n_i) \cdot (b \otimes \bar{b}) = \sum_i a m_i b \otimes_A \bar{a} n_i \bar{b}. \tag{8}$$

For any two A^e-rings (U, i), (V, j), $U \times_A V$ is an A^e-ring via the well defined algebra map $A \otimes \bar{A} \to U \times_A V$, $a \otimes \bar{b} \to i(a) \otimes_A j(\bar{b})$. Note that if (H, s, t) is an A^e-ring and H is considered as an A^e-bimodule via the actions described at the end of Section 2.2, then $H \times_A H = \Gamma(H, s, t)$.

For M, N and $P \in {}_{A^e}\mathcal{M}_{A^e}$ define

$$M \times_A P \times_A N := \int^{s,u} \int_{r,t} {}_{\bar{r}}M_{\bar{s}} \otimes {}_{r,\bar{t}}P_{s,\bar{u}} \otimes {}_tN_u.$$

There exist natural maps

$$\alpha : (M \times_A P) \times_A N \to M \times_A P \times_A N, \quad \alpha' : M \times_A (P \times_A N) \to M \times_A P \times_A N.$$

The maps α, α' are not isomorphisms in general. Since $\mathrm{End}(A)$ is an A^e-ring (cf. Section 2.2), it is an A^e-bimodule, so one can define the maps

$$\theta : M \times_A \mathrm{End}(A) \to M, \quad \theta(\sum_i m_i \otimes_A f_i) = \sum_i \overline{f_i(1)} m_i,$$

$$\theta' : \mathrm{End}(A) \times_A M \to M, \quad \theta'(\sum_i f_i \otimes_A m_i) = \sum_i f_i(1) m_i,$$

Following [19] a triple (L, Δ, μ) is called a $\times_A$-*coalgebra* iff L is an A^e-bimodule and $\Delta : L \to L \times_A L$, $\mu : L \to \mathrm{End}(A)$ are A^e-bimodule maps such that

$$\alpha \circ (\Delta \times_A L) \circ \Delta = \alpha' \circ (L \times_A \Delta) \circ \Delta, \quad \theta \circ (L \times_A \mu) \circ \Delta = L = \theta' \circ (\mu \times_A L) \circ \Delta.$$

Remark 2.5 ([15]). Let L be an A^e-bimodule and $\Delta : L \to L \times_A L$ and $\mu : L \to \mathrm{End}(A)$ A^e-bimodules maps. Let $i : L \times_A L \to L \otimes_A L$ be the canonical inclusion. Then (L, Δ, μ) is a $\times_A$-coalgebra if and only if $(L, \Delta', \epsilon_\mu)$ is an A-coring, where $\Delta' = i \circ \Delta$ and $\epsilon_\mu(l) = \mu(l)(1_A)$.

Definition 2.6 ([19]). Let (H, s, t) be an A^e-ring. (H, s, t, Δ, μ) is called a $\times_A$-*bialgebra* if (H, Δ, μ) is a $\times_A$-coalgebra and Δ and μ are maps of A^e-rings.

3. $\times_A$-bialgebras versus bialgebroids

The aim of this section is to clarify that the three notions recalled in the previous section are in fact equivalent to each other.

Theorem 3.1. *For an A^e-ring (H, s, t), the following data are equivalent :*

(1) A bialgebroid structure $(H, s, t, \Delta, \epsilon)$;

(2) A bialgebroid with an anchor structure (H, s, t, Δ, μ);

(3) A $\times_A$-bialgebra structure (H, s, t, Δ, μ);

(4) A monoidal structure on ${}_H\mathcal{M}$ such that the forgetful functor $F : {}_H\mathcal{M} \to {}_A\mathcal{M}_A$ is strict monoidal.

Proof. The equivalence (3) $\Leftrightarrow$ (4) is proven in [14, Theorem 5.1].

(1) $\Rightarrow$ (2), (3). Let $(H, s, t, \Delta, \epsilon)$ be an A-bialgebroid in the sense of Definition 2.2, and define (cf. [8, Eq. (8)])

$$\mu = \mu_\epsilon : H \to \mathrm{End}(A), \quad \mu(h)(a) = h \triangleright a := \epsilon(hs(a)) = \epsilon(ht(a)). \tag{9}$$

The map μ is an algebra morphism since $(A, \triangleright) \in {}_H\mathcal{M}$. The fact that μ is A-bilinear follows by an elementary calculation. Explicitly, for any a, $b \in A$, $h \in H$ we have

$$\mu(a \cdot h)(b) = \mu(s(a)h)(b) = \epsilon(s(a)hs(b)) = a\epsilon(hs(b)) = a\mu(h)(b) = (a \cdot \mu(h))(b),$$

where we used (1) to derive the last equality, thus proving that μ is left A-linear. Similar calculation that uses (6), proves the right A-linearity of μ. Next we prove that (A1) and (A2) hold for μ. Using (B2) and (5) we have $\Delta(hs(a)) = \Delta(h)\Delta(s(a)) = h_{(1)}s(a) \otimes_A h_{(2)}$. Now using the first part of the counit property (4) for $hs(a)$, we obtain $s\Big(\epsilon(h_{(1)}s(a))\Big)h_{(2)} = hs(a)$, i.e., (A1) for μ. The condition (A2) follows from $h \triangleright a = \epsilon(ht(a))$ and the second part of the counit property (4) together with (B2) and (5). This shows that $(H, s, t, \Delta, \mu = \mu_\epsilon)$ is an A-bialgebroid with an anchor in the sense of Definition 2.4, i.e., (1) $\Rightarrow$ (2).

In fact there is more, and this is (1) $\Rightarrow$ (3). Since μ and Δ are A-bimodule maps, they are left A^e-module maps. Furthermore, both μ and the corestriction Δ' of Δ to $\Gamma = H \times_A H$, are k-algebra maps. Therefore, by the observation at the end of

Section 2.1, μ and Δ' are maps of A^e-rings, and hence also maps of A^e-bimodules. Remark 2.5 then implies that $(H, s, t, \Delta', \mu = \mu_\epsilon)$ is a $\times_A$-bialgebra.

(2) $\Rightarrow$ (1). Let (H, s, t, Δ, μ) be a bialgebroid with an anchor and define $\epsilon = \epsilon_\mu : H \to A$, $h \mapsto \mu(h)(1_A)$. [20, Proposition 3.3] shows that ϵ is a map in ${}_A\mathcal{M}_A$, and a counit for Δ. This also implies that $\epsilon(s(a)) = \epsilon(t(a)) = a$, hence, in particular that $\epsilon(1_H) = 1_A$. Furthermore by [20, Proposition 3.5], for all $g, h \in H$, $\epsilon(gh) = \mu(g)(\epsilon(h))$. Therefore $\epsilon(gh) = \mu(g)(\epsilon(h)) = \mu(g)(\epsilon(s(\epsilon(h)))) = \epsilon(hs(\epsilon(h)))$, and similarly for the target map t. This proves equations (3), and we conclude that $(H, s, t, \Delta, \epsilon_\mu)$ is a bialgebroid in the sense of Definition 2.2.

The above result gives μ in terms of $\epsilon = \epsilon_\mu$. Looking at (9) one can apply (1) $\Rightarrow$ (2) once again. In this way one obtains that for a bialgebroid with an anchor (H, s, t, Δ, μ) the structure maps Δ' (the corestriction of Δ to $H \times_A H$) and μ are in fact maps of A^e-rings. Following Remark 2.5 the implication (3) $\Rightarrow$ (2) is then obvious. $\square$

4. Braided commutative algebras and bialgebroids

Let H be a bialgebra, $(A, \cdot)$ a left H-module algebra and let $A\#H = A \otimes H$ as a k-module with the multiplication

$$(a\#g)(b\#h) = a(g_{(1)} \cdot b)\#g_{(2)}h.$$

Let ${}_H\mathcal{YD}^H$ denote the pre-braided monoidal category of (left-right) crossed or Yetter-Drinfeld modules. This means that $(M, \cdot, \rho^M) \in {}_H\mathcal{YD}^H$ if and only if $(M, \cdot)$ is a left H-module, (M, ρ^M) is a right H-comodule and

$$h_{(1)} \cdot m_{<0>} \otimes h_{(2)}m_{<1>} = (h_{(2)} \cdot m)_{<0>} \otimes (h_{(2)} \cdot m)_{<1>}h_{(1)}, \tag{10}$$

for all $h \in H$, $m \in M$. For all M, $N \in {}_H\mathcal{YD}^H$, $M \otimes N \in {}_H\mathcal{YD}^H$ via

$$h \cdot (m \otimes n) = h_{(1)} \cdot m \otimes h_{(2)} \cdot n, \quad m \otimes n \to m_{<0>} \otimes n_{<0>} \otimes n_{<1>}m_{<1>}.$$

The pre-braiding (a braiding if H has an antipode) is given by

$$\sigma_{M,N} : M \otimes N \to N \otimes M, \quad \sigma_{M,N}(m \otimes n) = n_{<0>} \otimes n_{<1>} \cdot m.$$

An algebra A which is also an object in ${}_H\mathcal{YD}^H$ via $(A, \cdot, \rho^A)$, is called an *algebra in* ${}_H\mathcal{YD}^H$ if the algebra structures are maps in the category ${}_H\mathcal{YD}^H$. This is equivalent to say that $(A, \cdot)$ is a left H-module algebra and (A, ρ^A) is a right H^{op}-comodule algebra. An algebra A in ${}_H\mathcal{YD}^H$ is said to be *braided commutative* if the multiplication m_A of

A is commutative with respect to $\sigma_{A,A}$, i.e., $m_A \circ \sigma_{A,A} = m_A$, or equivalently, for all $a, b \in A$,

$$b_{<0>}(b_{<1>} \cdot a) = ab. \tag{11}$$

The following theorem is a generalisation of [8, Theorem 5.1].

Theorem 4.1. *Let H be a bialgebra, $(A, \cdot)$ a left H-module algebra and (A, ρ^A) a right H-comodule. Then $(A, \cdot, \rho^A)$ is a braided commutative algebra in ${}_H\mathcal{YD}^H$ if and only if $(A\#H, s, t, \Delta, \epsilon)$ is an A-bialgebroid with the source, target, comultiplication and the counit given by $s(a) = a\#1_H$, $t(a) = a_{<0>}\#a_{<1>}$, $\Delta(a\#h) = a\#h_{(1)} \otimes_A 1_A\#h_{(2)}$, $\epsilon(a\#h) = \epsilon_H(h)a$, for all $a \in A$ and $h \in H$.*

Furthermore, if H has an antipode S, then $A\#H$ is a Hopf algebroid with the antipode

$$\tau : A\#H \to A\#H, \quad \tau(a\#h) = \Big(S(h_{(2)})S^2(a_{<1>})\Big) \cdot a_{<0>}\#S(h_{(1)})S^2(a_{<2>})$$

for all $a \in A$ and $h \in H$.

Proof. Clearly, s is an algebra map. We prove now that t is an anti-algebra map if and only if (A, ρ^A) is a right H^{op}-comodule algebra and the braided commutativity relation (11) holds. Take any $a, b \in A$, then $t(ab) = (ab)_{<0>}\#(ab)_{<1>}$, and $t(b)t(a) = b_{<0>}(b_{<1>} \cdot a_{<0>})\#b_{<2>}a_{<1>}$. Suppose t is an anti-algebra map. Then applying $A \otimes \epsilon_H$ to the above equality one obtains equation (11). It follows then that $t(b)t(a) = a_{<0>}b_{<0>}\#b_{<1>}a_{<1>}$, i.e. $\rho^A : A \to A \otimes H^{op}$ is an algebra map, hence (A, ρ^A) is an H^{op}-comodule algebra as required. Conversely, suppose that (A, ρ^A) is a right H^{op}-comodule algebra and equation (11) holds. Then

$$t(b)t(a) = b_{<0>}(b_{<1>} \cdot a_{<0>})\#b_{<2>}a_{<1>} = a_{<0>}b_{<0>}\#b_{<1>}a_{<1>} = t(ab).$$

Assume now that t is an anti-algebra map. We prove that $\mathrm{Im}(\Delta) \subseteq \Gamma$ if and only if $(A, \cdot, \rho^A) \in {}_H\mathcal{YD}^H$. $A\#H$ is a right A-module via (2), i.e., using (11) we have

$$(b\#h) \cdot a = t(a)(b\#h) = a_{<0>}(a_{<1>} \cdot b)\#a_{<2>}h = ba_{<0>}\#a_{<1>}h. \tag{12}$$

$\mathrm{Im}(\Delta) \subseteq \Gamma$ if and only if for all $a, b \in A$, $h \in H$ we have that $(a\#h_{(1)})(b_{<0>}\#b_{<1>}) \otimes_A 1\#h_{(2)} = a\#h_{(1)} \otimes_A (1\#h_{(2)})(b\#1)$ or, equivalently,

$$a(h_{(1)} \cdot b_{<0>})\#h_{(2)}b_{<1>} \otimes_A 1\#h_{(3)} = a\#h_{(1)} \otimes_A h_{(2)} \cdot b\#h_{(3)}.$$

Since the tensor product is defined over A and equality (12) holds we have

$$a(h_{(1)}\cdot b_{<0>})\#h_{(2)}b_{<1>}\otimes_A 1\#h_{(3)} = a(h_{(2)}\cdot b)_{<0>}\#(h_{(2)}\cdot b)_{<1>}h_{(1)}\otimes_A 1\#h_{(3)}.$$

Thus we conclude that $\mathrm{Im}(\Delta)\subseteq\Gamma$ if and only if $h_{(1)}\cdot b_{<0>}\otimes h_{(2)}b_{<1>} = (h_{(2)}\cdot b)_{<0>}\otimes (h_{(2)}\cdot b)_{<1>}h_{(1)}$, i.e., if and only if $(A,\cdot,\rho^A)\in {}_H\mathcal{YD}^H$.

Therefore we have proven that t is an anti-algebra map and $\mathrm{Im}(\Delta)\subseteq\Gamma$ if and only if $(A,\cdot,\rho^A)$ is a braided commutative algebra in ${}_H\mathcal{YD}^H$. It is then straightforward to check that all the remaining conditions in Definition 2.2 hold.

Finally we prove that τ defined in the theorem is the antipode of $A\#H$. The canonical projection $(A\#H)\otimes(A\#H)\to(A\#H)\otimes_A(A\#H)$ has a well defined section $\gamma : (A\#H)\otimes_A(A\#H)\to(A\#H)\otimes(A\#H)$, $\gamma(a\#h\otimes_A b\#g) = ab_{<0>}\#b_{<1>}h\otimes 1_A\#g$. Since for all $a\in A$, $h\in H$, $\tau(1_A\#h) = 1_A\#S(h)$ and $\tau(a\#1_H) = S^2(a_{<1>})\cdot a_{<0>}\#S^2(a_{<2>})$, we have

$$\tau(a\#h) = \tau((a\#1_H)(1_A\#h)) = \tau(1_A\#h)\tau(a\#1_H),$$

i.e., τ is an anti-algebra map. Condition (ANT3) follows from the definition of γ,

$$(a\#h_{(1)})\tau(1_A\#h_{(2)}) = (a\#h_{(1)})(1_A\#S(h_{(2)})) = \epsilon_H(h)a\#1_H,$$

while (ANT1) can be established by the following computation

$$\begin{aligned}
\tau(t(a)) &= \tau(a_{<0>}\#a_{<1>}) = S(a_{<4>})S^2(a_{<1>})\cdot a_{<0>}\#S(a_{<3>})S^2(a_{<2>})\\
&= S(a_{<4>})S^2(a_{<1>})\cdot a_{<0>}\#S\Big(S(a_{<2>})a_{<3>}\Big)\\
&= S\Big(S(a_{<1>})a_{<2>}\Big)\cdot a_{<0>}\#1_H = a\#1_H = s(a).
\end{aligned}$$

It remains to prove property (ANT2). The left hand side of (ANT2) equals

$$\tau(a\#h_{(1)})(1_A\#h_{(2)}) = S(h_{(2)})S^2(a_{<1>})\cdot a_{<0>}\#S(h_{(1)})S^2(a_{<2>})h_{(3)}.$$

Equation (10), evaluated at $\Delta(S(h_{(1)}))\otimes h_{(2)} = S(h_{(2)})\otimes S(h_{(1)})\otimes h_{(3)}$ implies that

$$Sh_{(2)}\cdot a_{<0>}\otimes Sh_{(1)}a_{<1>}h_{(3)} = \Big(S(h)\cdot a_{<0>}\Big)_{<0>}\otimes\Big(S(h)\cdot a_{<0>}\Big)_{<1>} \tag{13}$$

for all $h \in H$ and $a \in A$. Now the right hand side of (ANT2) reads

$$\begin{aligned}
(t\circ\epsilon\circ\tau)(a\#h) &= t(S(h)S^2(a_{<1>})\cdot a_{<0>}) \\
&= \Big(S(S(a_{<1>})h)\cdot a_{<0>}\Big)_{<0>}\#\Big(S(S(a_{<1>})h)\cdot a_{<0>}\Big)_{<1>} \\
(13)\qquad &= S((S(a_{<1>})h)_{(2)})\cdot a_{<0>}\#S((S(a_{<1>})h)_{(1)})a_{<1>}(S(a_{<1>})h)_{(3)} \\
&= S(h_{(2)})S^2(a_{<3>})\cdot a_{<0>}\#S(h_{(1)})S^2(a_{<4>})a_{<1>}S(a_{<2>})h_{(3)} \\
&= S(h_{(2)})S^2(a_{<1>})\cdot a_{<0>}\#S(h_{(1)})S^2(a_{<2>})h_{(3)}
\end{aligned}$$

that is exactly the left hand side of (ANT2). Hence, τ is an antipode of $A\#H$. □

Theorem 4.1 generalises, gives converse to, and a more transparent proof of [8, Theorem 5.1]. It also provides one with a rich source of examples of bialgebroids. Several examples of braided commutative algebras in ${}_H\mathcal{YD}^H$ are known, cf. [4], [5], [9]. For example, for an H^{op}-Galois extension, A/B, the centralizer algebra $E = C_A(B)$ has a structure of a braided commutative algebra in ${}_H\mathcal{YD}^H$ ([4]).

We indicate now three other ways of obtaining braided commutative algebras.

Example 4.2. 1. Let $(H, R = \sum R^1\otimes R^2)$ be a quasitriangular bialgebra and $(A,\cdot)$ a left H-module algebra, which is braided commutative in the pre-braided category ${}_H\mathcal{M}$: i.e. $\sum(R^2\cdot b)(R^1\cdot a) = ab$, for all a, $b\in A$. Then A is a braided commutative algebra in ${}_H\mathcal{YD}^H$ where the coaction of H on A is given by $\rho^A : A\to A\otimes H$, $a\mapsto \sum R^2\cdot a\otimes R^1$. In this way, all examples of braided commutative algebras over a quasi-triangular Hopf algebra H from [5] give examples of braided commutative algebras in ${}_H\mathcal{YD}^H$, and hence examples of bialgebroids $A\#H$. Note that [8, Theorem 5.1] corresponds to the quasi-triangular Hopf algebra $(D(H),\mathcal{R})$.

2. Dually, let (H,σ) be a coquasitriangular bialgebra and (A,ρ^A) be a right H^{op}-comodule algebra such that $ab = \sigma(a_{<1>}\otimes b_{<1>})b_{<0>}a_{<0>}$, for all a, $b\in A$. Then $(A,\cdot,\rho^A)$ is a braided commutative algebra in ${}_H\mathcal{YD}^H$, where the left H-action is $h\cdot a = \sigma(a_{<1>}\otimes h)a_{<0>}$.

3. There is a general way of constructing braided commutative algebras in ${}_H\mathcal{YD}^H$ pointed out in [9], [4]. Let $(V,\cdot,\rho^V)\in {}_H\mathcal{YD}^H$ and $T(V)$ be the tensor algebra of V. Then the (co)-actions of H on V extend uniquely to (co)-actions on $T(V)$ such that $T(V)$ becomes an algebra in the category ${}_H\mathcal{YD}^H$. Let $S^b(V)$ be the "braided symmetric" algebra of V, i.e., $S^b(V) := T(V)/I$, where I is the two-sided ideal of

$T(V)$ generated by all elements of the form $v \otimes w - w_{<0>} \otimes w_{<1>} \cdot v$, for all $v, w \in V$. Then $S^b(V)$ is a braided commutative algebra in ${}_H\mathcal{YD}^H$.

Using the FRT-construction and Example 4.2 we present now a generic construction of bialgebroids associated to any solution of the quantum Yang-Baxter equation.

Let n be a positive integer and $R = (R^{ij}_{uv}) \in M_n(k) \otimes M_n(k)$ be a solution of the QYBE, $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$. Let $A(R)$ be the bialgebra associated to R using the FRT construction: $A(R)$ is a free k-algebra generated be $(c^i_j)_{i,j=1,\cdots,n}$ with the relations $R^{ij}_{vu}c^u_k c^v_l = R^{vu}_{lk}c^i_v c^j_u$, for all i, j, k, $l = 1, \cdots, n$ (Einstein's summation convention assumed), and the standard matrix bialgebra structure. View R as an endomorphism of $V \otimes V$, for an n-dimensional vector space, and define the corresponding braided symmetric algebra $S^b(V) = S_R(n)$ as follows. $S_R(n)$ is a free k-algebra generated by $\xi_1, \cdots, \xi_n$ with the relations $\xi_u\xi_v = R^{li}_{uv}\xi_i\xi_l$, for all u, $v = 1, \cdots, n$. $S_R(n)$ is a braided commutative algebra in ${}_{A(R)}\mathcal{YD}^{A(R)}$ via $c^j_v \cdot \xi_u = R^{ij}_{uv}\xi_i$ and $\xi_v \to \xi_u \otimes c^u_v$, for all j, u, $v = 1, \cdots, n$. Then Theorem 4.1 implies the following example of a quantum groupoid

Proposition 4.3. *Let n be a positive integer and $R = (R^{ij}_{uv})$ a solution of the QYBE. Then the smash product $S_R(n)\#A(R)$ has a structure of $S_R(n)$-bialgebroid with the source, target, comultiplication and the counit given by $s(\xi_i) = \xi_i\#1$, $t(\xi_i) = \xi_u\#c^u_i$, $\Delta(\xi_i\#c^u_v) = \xi_i\#c^u_l \otimes_{S_R(n)} 1\#c^l_v$, and $\epsilon(\xi_i\#c^u_v) = \delta_{u,v}\xi_i$, where $i, u, v = 1, \cdots, n$.*

In particular, Proposition 4.3 associates bialgebroids to quantum matrix groups such as $GL_q(n)$ and their corepresentation spaces such as the quantum hyperplane (cf. [10]).

5. Comments on duals of bialgebroids - bicoalgebroids

On formal level, the notion of a bialgebra is self-dual in the following sense. Write definition of a bialgebra in terms of commutative diagrams. Then the structure obtained by reversing arrows in diagrams defining a bialgebra is again a bialgebra. It is clear that, in general, a dual of a bialgebroid in the above sense is no longer a bialgebroid.[1] This is because by reversing the arrows in diagrams defining an algebra and a module one obtains diagrams defining a coalgebra and a comodule. Thus if one wants

[1]This should not be confused with a left or right dual module of a bialgebroid which is a bialgebroid provided certain finitely generated projective type conditions are satisfied (cf. [7]).

to construct a (formally) dual object to a bialgebroid one has to consider an object within the category of comodules of a coalgebra.

Definition 5.1. Let C be a coalgebra over a field k. A *bicoalgebroid* is a k-coalgebra H which satisfies the following conditions:

(BC1) There is a coalgebra map $\alpha : H \to C$ and an anti-coalgebra map $\beta : H \to C$ such that for all $h \in H$, $\alpha(h_{(1)}) \otimes \beta(h_{(2)}) = \alpha(h_{(2)}) \otimes \beta(h_{(1)})$. This allows one to view H as a C-bicomodule via left coaction ${}^H\rho(h) = \alpha(h_{(1)}) \otimes h_{(2)}$, and the right coaction $\rho^H(h) = h_{(2)} \otimes \beta(h_{(1)})$. Let

$$H\Box_C H = \{\sum_i g^i \otimes h^i \in H \otimes H \mid \sum_i g^i{}_{(2)} \otimes \beta(g^i{}_{(1)}) \otimes h^i = \sum_i g^i \otimes \alpha(h^i{}_{(1)}) \otimes h^i{}_{(2)}\}$$

be the corresponding cotensor product.

(BC2) There is a C-bicomodule map $\mu : H\Box_C H \to H$ which is an associative product with respect to the cotensor product and such that for all $\sum_i g^i \otimes h^i \in H\Box_C H$:
   (a) $\sum_i \mu(g^i \otimes h^i{}_{(1)}) \otimes \alpha(h^i{}_{(2)}) = \sum_i \mu(g^i{}_{(1)} \otimes h^i) \otimes \beta(g^i{}_{(2)})$,
   (b) $\Delta(\mu(\sum_i g^i \otimes h^i)) = \sum_i \mu(g^i{}_{(1)} \otimes h^i{}_{(1)}) \otimes \mu(g^i{}_{(2)} \otimes h^i{}_{(2)})$.

(BC3) There exists a bicomodule map $\eta : C \to H$ which is a unit for μ, i.e.,

$$\mu \circ (\eta\Box_C H) \circ {}^H\rho = \mu \circ (H\Box_C \eta) \circ \rho^H = H,$$

and such that for all $c \in C$, $\epsilon(\eta(c)) = \epsilon(c)$, and

$$\Delta(\eta(c)) = \eta(c)_{(1)} \otimes \eta(\alpha(\eta(c)_{(2)})) = \eta(c)_{(1)} \otimes \eta(\beta(\eta(c)_{(2)})).$$

Few comments are needed in order to see that the above definition makes sense. The condition (BC2)(a) makes sense because for all $\sum_i g^i \otimes h^i \in H\Box_C H$ we have

$$\sum_i g^i{}_{(2)} \otimes g^i{}_{(3)} \otimes \beta(g^i{}_{(1)}) \otimes h^i = \sum_i g^i{}_{(1)} \otimes g^i{}_{(2)} \otimes \alpha(h^i{}_{(1)}) \otimes h^i{}_{(2)}, \tag{14}$$

$$\sum_i g^i{}_{(2)} \otimes \beta(g^i{}_{(1)}) \otimes h^i{}_{(1)} \otimes h^i{}_{(2)} = \sum_i g^i \otimes \alpha(h^i{}_{(1)}) \otimes h^i{}_{(2)} \otimes h^i{}_{(3)} \tag{15}$$

Equation (14) implies that $\sum_i g^i{}_{(1)} \otimes h^i \otimes g^i{}_{(2)} \in H\Box_C H \otimes H$ while Equation (15) implies that $\sum_i g^i \otimes h^i{}_{(1)} \otimes h^i{}_{(2)} \in H\Box_C H \otimes H$. Furthermore both equations (14) and (15) imply that $\sum_i g^i{}_{(1)} \otimes h^i{}_{(1)} \otimes g^i{}_{(2)} \otimes h^i{}_{(2)} \in H\Box_C H \otimes H \otimes H$. Using condition (BC2)(a) one concludes that $\sum_i \mu(g^i{}_{(1)} \otimes h^i{}_{(1)}) \otimes g^i{}_{(2)} \otimes h^i{}_{(2)} \in H \otimes H\Box_C H$, i.e., condition (BC2)(b) makes sense. Note that conditions (BC2) and (BC3) mean also that H is a C-ring in the sense of [3, Section 6].

One way of understanding the relation of an object defined in Definition 5.1 to bialgebroids is to write all the conditions in terms of commutative diagrams. Reversing the arrows, replacing C by A, α by s, β by t, $\Box_C$ by $\otimes_A$, μ by Δ and η by ϵ one obtains commutative diagrams defining a bialgebroid.

An indication that this dualisation of a bialgebroid might play a role in introducing self-dual bialgebroids comes from the following observation. A self-dual generalisation of a Hopf algebra is provided by the notion of a *weak Hopf algebra* [1]. A *weak bialgebra* is a vector space H which is an algebra and a coalgebra with multiplicative (but non-unital) coproduct such that for all $x, y, z \in H$, $\epsilon(xyz) = \epsilon(xy_{(1)})\epsilon(y_{(2)}z) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z)$, and $(\Delta \otimes H) \circ \Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$. A weak bialgebra H is a *weak Hopf algebra* if there exists an antipode, i.e., a linear map $S : A \to A$ such that for all $h \in H$, $h_{(1)}S(h_{(2)}) = \epsilon(1_{(1)}h)1_{(2)}$, $S(h_{(1)})h_{(2)} = 1_{(1)}\epsilon(h1_{(2)})$, and $S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h)$. Weak Hopf algebras have been studied in connection to integrable models and classification of subfactors of von Neumann algebras. In [6, Proposition 2.3.1] it has been shown that a weak Hopf algebra with bijective antipode is a bialgebroid over $A = \mathrm{Im}\epsilon_t$, where $\epsilon_t : H \to H$, $h \mapsto \epsilon(1_{(1)}h)1_{(2)}$. By [1, Eq. (2.12)] $\ker \epsilon_t$ is a coideal, hence we can state the following

Proposition 5.2. *Let H be a weak Hopf algebra with bijective antipode. Let $C = H/\ker \epsilon_t$ with the canonical surjection $\pi_t : H \to C$. Then C is a coalgebra and H is a bicoalgebroid over C with the following structure maps:*

1. $\alpha = \pi_t$, $\beta = \pi_t \circ S^{-1}$,
2. $\mu(\sum_i g^i \otimes h^i) = \sum_i g^i h^i$, *for all* $\sum_i g^i \otimes h^i \in H \Box_C H$.
3. $\eta : C \to H$, $c \mapsto \epsilon_t(h)$, *where* $h \in \pi_t^{-1}(c)$.

Proof. This can be proven by dualising the proof of [6, Proposition 2.3.1]. Although not elementary this is quite straightforward and we leave it to the reader[2]. □

Thus a weak Hopf algebra is an example of both a bialgebroid and a bicoalgebroid. This suggests that if one imposes a selfduality as a key property that must be enjoyed by a proper generalisation of a bialgebra, such a generalisation should be a bialgebroid over an algebra A and a bicoalgebroid over a coalgebra C at the same time. Some

[2]The authors will be happy to supply the full proof upon request.

relations between A and C should also be required in order to compare tensor products with cotensor products. Once such a relationship is imposed compatibility conditions between product and coproduct must involve both tensor and cotensor products. What these should be we consider an interesting open question.

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